hep-th/0701028
MCTP-07-01
TAUP-2844-07

Finite Heisenberg Groups from Nonabelian Orbifold Quiver Gauge Theories

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Abstract

A large class of orbifold quiver gauge theories admits the action of finite Heisenberg groups of the form $\prod_i \mathrm{Heis}(\mathbb{Z}_{q_i} \times \mathbb{Z}_{q_i})$. For an Abelian orbifold generated by Γ, the $\mathbb{Z}_{q_i}$ shift generator in each Heisenberg group is one cyclic factor of the Abelian group Γ. For general non-Abelian Γ, however, we find that the shift generators are the cyclic factors in the Abelianization of Γ. We explicitly show this for the case $\Gamma = \Delta(27)$, where we construct the finite Heisenberg group symmetries of the field theory. These symmetries are dual to brane number operators counting branes on homological torsion cycles, which therefore do not commute. We compare our field theory results with string theory states and find perfect agreement.

1 Introduction

The AdS/CFT correspondence, which relates field theories with dual string (gravitational) theories, is well on its way to becoming firmly established as an especially important new tool for investigating field theories outside of perturbative control. As a strong/weak coupling duality, AdS/CFT and its variants have opened up a whole new window into the exploration of strongly coupled gauge theories through their weakly coupled string counterparts. Certainly, much of the focus of gauge/string duality has been on the use of appropriate string duals as a way to address important non-perturbative issues in gauge theories. However, given the limited technology for studying string theory in non-trivial closed string backgrounds (*i.e.* curved backgrounds with RR flux), one may ask whether it is possible to use AdS/CFT in the other direction to explore the structure of string theory in such backgrounds starting from our understanding of gauge theories.

In fact, Gukov, Rangamani and Witten in [1] did just this; based on properties of the dual orbifold quiver gauge theory, they found the novel feature that brane number charges counting branes wrapped on torsion cycles[1] do not commute. In particular, they examined the duality between string theory on $\mathrm{AdS}_5 \times S^5/\mathbb{Z}_3$ and the quiver gauge theory corresponding to a stack of D3-branes placed at the singular point of $\mathbb{R}^{1,3} \times \mathbb{C}^3/\mathbb{Z}_3$, and demonstrated that the resulting quiver gauge theory admits a set of discrete global symmetries that forms a Heisenberg group. Based on this, they concluded that, on the gravitational side, F-string and D-string number operators do not commute when the strings are wrapped on torsion cycles. In fact, these operators close on the number operator for D3-branes wrapping a torsion 3-cycle, and it is this D3-brane number operator which plays the part of the central extension in the Heisenberg group.

A similar observation about fluxes on torsion cycles was recently made by Freed, Moore and Segal in [2, 3] for generalized U(1) gauge theories. In this case, Poincaré duality of the theory requires that two gradings (electric and magnetic) of the Hilbert space be possible. Furthermore, in the presence of torsion cycles, these two gradings cannot be simultaneously implemented: *i.e.* electric and magnetic charges do not commute. As argued in [2], it is precisely this effect that is responsible for the non-commutativity of charges found in [1].

The non-commutativity in [1] was found as a non-commutativity of global symmetries of the quiver gauge theory dual. These discrete symmetries fall into two classes,

[1]Here we will always refer to the topological "torsion" groups appearing in integer valued homology rather than notions relating to modifications of connections on a tangent bundle. To stress this, we will usually refer to these topological cycles as "torsion cycles."

with the first being permutation type symmetries that map gauge groups into gauge groups (and correspondingly bifundamentals into bifundamentals) and the second being rephasing symmetries which act on the links in such a manner that all trace type gauge invariant composite operators remain unchanged under the rephasing. We will refer to the permutation symmetries here as $\mathcal{A}$ type symmetries and the rephasing symmetries as either $\mathcal{B}$ (if they do not commute with permutations) or $\mathcal{C}$ (if they are central) type symmetries.

In particular, for the $S^5/\mathbb{Z}_3$ quiver theory considered in [1], the permutation symmetries are generated by a 'shift' operator $\mathcal{A}$ of order three which cyclically permutes the three nodes of the quiver. (Note that this operation is simply the action of the $\mathbb{Z}_3$ orbifold group on the quiver itself.) In addition, the rephasing symmetries are generated by an order three 'clock' operator $\mathcal{B}$. Taken together, $\mathcal{A}$ and $\mathcal{B}$ do not commute, and form the Heisenberg group $\mathrm{Heis}(\mathbb{Z}_3 \times \mathbb{Z}_3)$ according to

$$\mathcal{A}^3 = \mathcal{B}^3 = \mathcal{C}^3 = 1, \qquad \mathcal{AB} = \mathcal{BAC}, \tag{1.1}$$

where $\mathcal{C}$ is a central element associated with permutation invariant rephasings.

The finite Heisenberg group identified above is the group of discrete global symmetries acting on the quiver gauge theory. However, as argued in [1], these symmetries ought to persist in the dual description of strings on $\mathrm{AdS}_5 \times S^5/\mathbb{Z}_3$. In this dual string picture, the $\mathcal{A}$ operator is mapped to the number operator associated with wrapped F-strings, while the $\mathcal{B}$ and $\mathcal{C}$ operators correspond to number operators associated with wrapped D-strings and D3-branes, respectively. Although S^5 by itself admits no non-trivial cycles, $S^5/\mathbb{Z}_3$ admits the torsion cycles $H_1(S^5/\mathbb{Z}_3;\mathbb{Z}) = \mathbb{Z}_3$ and $H_3(S^5/\mathbb{Z}_3;\mathbb{Z}) = \mathbb{Z}_3$, which are precisely what are needed for strings and D3-branes to wrap. In this case, the non-commutativity of (1.1) is a consequence of the non-commutativity of fluxes [2, 3] associated with these cycles. Furthermore, the fact that trace type gauge invariant operators are inert under the $\mathcal{B}$ and $\mathcal{C}$ symmetries indicate that their dual operators affect only states that are non-perturbative in the $1/N$ expansion, giving additional justification to their association with D-branes.

The work of [1] was recently extended by studying other orbifold theories [4], the inclusion of fractional branes [5], and the effect of Seiberg duality on these global symmetries [6]. In all these investigations, however, the orbifolds under consideration were Abelian, being $\mathbb{Z}_n$ orbifolds of either S^5, $T^{1,1}$ or $Y^{p,q}$. For the case without fractional branes, the resulting discrete symmetry groups were all found to be isomorphic to $\mathrm{Heis}(\mathbb{Z}_n \times \mathbb{Z}_n)$, which is exactly what one would expect to have been inherited from the $\mathbb{Z}_n$ action of the orbifold group.

Although the bulk of the work on orbifold models focuses on the Abelian case, in general non-Abelian orbifolds may also be constructed. Other than for added technical issues which arise in the latter case, there is no fundamental distinction between Abelian and non-Abelian orbifolds. Hence it is natural to expect that the identification of discrete global symmetries of Abelian orbifold quiver gauge theories carries over to non-Abelian orbifolds as well. Demonstrating that this is in fact the case will be the focus of the present paper. In particular, we examine quiver gauge theories that are dual to IIB string theory on $\mathrm{AdS}_5 \times S^5/\Gamma$, where now Γ is some non-Abelian discrete subgroup of SU(3), so that the gauge theory preserves at least $\mathcal{N} = 1$ supersymmetry. Although we only focus on orbifolds of the five-sphere, it is in principle possible to use the same techniques for field theories dual to orbifolds of other spaces which admit isometries. One simply has to use the regular representation of the group Γ to act on the gauge indices, which is the natural way of making a "geometric" orbifold [7,8]. In the following, we will *always* be using the regular representation of Γ as the group acting on the gauge indices.

In the case where Γ is Abelian, the $\mathcal{A}$ type permutation symmetries are easy to visualize geometrically, as they simply correspond to the action of the group Γ mapping the image D3-branes to each other (so that F-strings stretched between D3-branes and their images close up to an operation of $\mathcal{A}$). This immediately suggests that the group of $\mathcal{A}$ type symmetries remains Γ, even when extended to the non-Abelian case. As we show below, however, this identification is not entirely correct; instead, the actual group generated by the $\mathcal{A}$ type symmetries turns out to be the Abelianization of Γ, which we denote by $\bar{\Gamma} \equiv \Gamma/[\Gamma, \Gamma]$.

Unlike the $\mathcal{A}$ type symmetries, which are conceptually easy to visualize based on the action of Γ on the quiver, the $\mathcal{B}$ and $\mathcal{C}$ type rephasing symmetries are more difficult to identify. Motivated by [1] as well as the dual string picture (where $\mathcal{A}$ and $\mathcal{B}$ are related by interchanging F-strings and D-strings), it is natural to identify the group of $\mathcal{B}$ type symmetries with $\bar{\Gamma}$ as well. However, the explicit identification of the rephasing symmetries is significantly different, and rather more intricate to work out. Nevertheless, as demonstrated in various examples, this general picture of $\mathcal{A}$ and $\mathcal{B}$ type operations turns out to be correct. The $\mathcal{A}$ and $\mathcal{B}$ symmetries do not commute, and instead close on a set of central elements given by $\mathcal{C}$. Since $\mathcal{A}$ and $\mathcal{B}$ are both identified with $\bar{\Gamma}$, the group formed by these symmetries is just the finite Heisenberg group

$$\mathrm{Heis}(\bar{\Gamma}, \bar{\Gamma}). \tag{1.2}$$

This is identified as the group of discrete global symmetries of the S^5/Γ orbifold quiver gauge theory.

Turning now to the dual string picture, we expect that the above quiver gauge theory may be obtained by placing a stack of N D3-branes near the orbifold singularity of $\mathbb{R}^{1,3} \times \mathbb{C}^3/\Gamma$. Since the near-horizon space is $\mathrm{AdS}_5 \times S^5/\Gamma$, in this dual picture the Heisenberg group symmetries (1.2) arise from the properties of strings and D3-branes wrapping appropriate cycles on S^5/Γ. Since the homology of S^5/Γ is given by [9]

$$H_1(S^5/\Gamma;\mathbb{Z}) = \bar{\Gamma}, \qquad H_3(S^5/\Gamma;\mathbb{Z}) = \bar{\Gamma}, \tag{1.3}$$

we indeed identify the appropriate cycles for F-strings, D-strings and D3-branes to wrap. In particular, since the string duals of $\mathcal{A}$ and $\mathcal{B}$ are the number operators counting wrapped F-strings and D-strings, they both must independently form groups isomorphic to H_1, which as we see is just $\bar{\Gamma}$. This of course agrees with the quiver result.

The final part of the duality picture is to identify $\mathcal{C}$ with wrapped D3-branes, and hence the group generated by $\mathcal{C}$ with H_3 (which again is just $\bar{\Gamma}$). This now allows us to make a stronger statement on the form of the Heisenberg group (1.2). Since $\bar{\Gamma}$ is Abelian, it may be decomposed into a set of cyclic groups

$$\bar{\Gamma} = \mathbb{Z}_{a_1} \otimes \mathbb{Z}_{a_2} \otimes \cdots, \tag{1.4}$$

where each factor is associated with a torsion cycle in $H_1(S^5/\Gamma)$. Because $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are all identified with $\bar{\Gamma}$, the central extension implicit in (1.2) may be more explicitly stated in the decomposition

$$\mathrm{Heis}(\bar{\Gamma},\bar{\Gamma}) = \mathrm{Heis}(\mathbb{Z}_{a_1},\mathbb{Z}_{a_1}) \otimes \mathrm{Heis}(\mathbb{Z}_{a_2},\mathbb{Z}_{a_2}) \otimes \cdots. \tag{1.5}$$

Each individual factor in this decomposition is connected to the non-commutativity of fluxes on that particular torsion cycle [2, 3]. (Since we take Γ to be a discrete subgroup of SU(3), there are only a limited number of torsion cycles that may arise in practice.) Our main result can then be stated as:

> Quiver gauge theories obtained as worldvolume theories on a stack of N D3-branes placed at the singularity of $\mathbb{C}^3/\Gamma$ where Γ is a (possibly non-Abelian) finite discrete subgroup of SU(3) admit the action of global symmetries generating a group of the form (1.5) where the factors $\mathbb{Z}_{a_i}$ are given by the Abelianization of Γ as in (1.4).

The structure of the paper is as follows. In section 2 we review the construction of quiver gauge theories obtained as the worldvolume theory on a stack of N D3-branes placed at the conical singularity of $\mathbb{C}^3/\Gamma$, paying particular attention to the case where Γ is non-Abelian. This section also presents a general construction of the set of permutations $\mathcal{A}$ and rephasing symmetries $\mathcal{B}$ and $\mathcal{C}$. Section 3 describes how the field theory results match those of the dual string theory states. Section 4 contains a number of explicit examples motivating our claims. Finally, in section 5, we present our conclusions and point out some open questions. We relegate explicit details of the Abelianization of discrete subgroups of SU(3) to Appendix A and the representations of $\Delta(27)$ to Appendix B.

2 Discrete symmetries of orbifold quiver gauge theories

Before turning to the construction of the discrete symmetry operations $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, we first summarize the basic features of $\mathcal{N}=1$ quiver gauge theories corresponding to a stack of N D3-branes at the singular point of the $\mathbb{C}^3/\Gamma$ orbifold. We are, of course, especially interested in the case where Γ is non-Abelian. Pioneering work towards constructing these orbifold theories began with [10–12], and subsequently the quivers were more fully developed in [7, 8].

2.1 $\mathcal{N}=1$ gauge theories from $\mathbb{C}^3/\Gamma$ orbifolds

From our point of view, an $\mathcal{N}=1$ quiver is essentially a set of nodes representing gauge groups and links which describe chiral multiplets transforming in the bifundamental representation. For a discrete orbifold group $\Gamma \subset \mathrm{SU}(3)$, the nodes of the quiver may be placed in one-to-one correspondence with the irreducible representations $\mathbf{r}_i$ of Γ, and the corresponding gauge groups are taken to be $\mathrm{SU}(n_i N)$ where $n_i = \dim(\mathbf{r}_i)$. In order to obtain the bifundamental matter, we must choose an appropriate embedding of the orbifold action in $\mathbb{C}^3$, corresponding to a (possibly reducible) three-dimensional representation $\mathbf{3}$ of Γ. The number of bifundamentals stretching from node i to node j is then given by b_{ij} in the decomposition

$$\mathbf{3} \otimes \mathbf{r}_i = \oplus_j b_{ij} \mathbf{r}_j. \tag{2.1}$$

From this definition, it is clear that b_{ij} is related to the familiar adjacency matrix via

$$a_{ij} = b_{ij} - b_{ji}. \tag{2.2}$$

Although not directly encoded in the quiver diagram itself, the (cubic) superpotential may be fully determined from the properties of the orbifold group Γ. In the notation of [8], the superpotential is given by

$$W = \sum h_{ijk}^{f_{ij}, f_{jk}, f_{ki}} \mathrm{Tr} \left(\Phi^{ij}_{f_{ij}} \Phi^{jk}_{f_{jk}} \Phi^{ki}_{f_{ki}} \right), \tag{2.3}$$

where

$$h_{ijk}^{f_{ij},f_{jk},f_{ki}} = \epsilon_{\alpha\beta\gamma}(Y_{f_{ij}})^{\alpha}_{v_i\overline{v}_j}(Y_{f_{jk}})^{\beta}_{v_j\overline{v}_k}(Y_{f_{ki}})^{\gamma}_{v_k\overline{v}_i}, \tag{2.4}$$

and $(Y_{f_{ij}})^{\alpha}_{v_i\overline{v}_j}$ is the Clebsch-Gordan coefficient corresponding to the decomposition of (2.1). Note that $f_{ij} = 1, \ldots, b_{ij}$ labels the specific link showing up in the above decomposition.

2.2 Discrete symmetries from one dimensional representations of Γ

Given a quiver theory constructed as above, our goal is now to identify potential discrete global symmetries of the quiver. These symmetries fall naturally into two categories:

1. Permutation symmetries which maps fields to fields and gauge groups to gauge groups. Motivated by the notation of [1], we label these permutations as $\mathcal{A}$ type symmetries.

2. Symmetries that rephase fields in such a way as to leave all trace type gauge invariant operators inert. These rephasing symmetries may be thought of as anomaly free discrete subgroups of the global U(1) symmetries acting at each node of the quiver. These symmetries will be denoted as either $\mathcal{B}$ type if they do not commute with the permutations or $\mathcal{C}$ type if they do.

2.2.1 Permutation symmetries

For a given quiver, the $\mathcal{A}$ type permutation symmetries are easy to visualize. If we view the quiver as a directed graph, then the $\mathcal{A}$ type symmetries correspond to the subgroup of the automorphism group of the quiver that leaves the superpotential invariant. In order to ensure that this is a symmetry of the full theory, it is important that the superpotential remains invariant. However, since this superpotential data is not manifest from the quiver diagram itself, additional input must be considered when determining the group of permutations generated by $\mathcal{A}$. As will be seen in the examples below, this group could be substantially smaller than the full automorphism group of the quiver diagram.

In order to systematically construct the $\mathcal{A}$ type symmetries, we first note that good permutations can only map amongst gauge groups of the same rank. Since each gauge group (node in the quiver) is labeled by an irreducible representation $\mathbf{r}_i$ of Γ, this indicates that good permutations must only map irreducible representations of the same dimension into each other. Such transformations are in fact naturally furnished by the one-dimensional representations of Γ, which we denote by $\mathbf{1}_\alpha$, where α labels the particular representation. The action of $\mathbf{1}_\alpha$ on the nodes of the quiver follows directly from the tensor product map

$$\mathbf{1}_\alpha \otimes \mathbf{r}_i = \mathbf{r}_{\alpha(i)}. \tag{2.5}$$

Note that, so long as $\mathbf{r}_i$ is irreducible, then so is $\mathbf{r}_{\alpha(i)}$. This may easily be seen because multiplication by the one-dimensional representation on the left does not affect how one may or may not block diagonalize a given matrix representation. As a result, the map (2.5) indeed gives rise to a good permutation on the nodes of the quiver, taking node i to node $\alpha(i)$.

What remains to be demonstrated is that the transformation induced by multiplication with $\mathbf{1}_\alpha$ also yields a permutation of the bifundamental links consistent with invariance of the superpotential. To see that this is the case, we recall that all superpotential terms in the orbifolded theory that are mapped to each other by the action of $\mathbf{1}_\alpha$ descend from a single superpotential term in the original un-orbifolded parent $\mathcal{N} = 4$ theory. This turns out to be sufficient to guarantee that the above action of the one-dimensional representations is a symmetry of the orbifold quiver.

To see this, we first examine the permutation of the links induced by $\mathbf{1}_\alpha$. Here we may simply follow the decomposition rules for the tensor product

$$\mathbf{3} \otimes \mathbf{r}_{\alpha(i)} = \mathbf{3} \otimes (\mathbf{1}_\alpha \otimes \mathbf{r}_i) = \mathbf{1}_\alpha \otimes (\mathbf{3} \otimes \mathbf{r}_i) = \oplus_j b_{ij} \mathbf{1}_\alpha \otimes \mathbf{r}_j = \oplus_j b_{ij} \mathbf{r}_{\alpha(j)}, \tag{2.6}$$

which demonstrates that the matrices b_{ij} before and after the transformation are related by

$$b_{\alpha(i)\alpha(j)} = b_{ij}, \tag{2.7}$$

and therefore so are the adjacency matrices. Since $\alpha(i)$ is just the relabeling of node i, this adjacency matrix relation is precisely what is desired for generating good permutations of the links.

Of course, we must also ensure that the superpotential remains invariant under the mapping induced by $\mathbf{1}_\alpha$. By examining (2.3) and (2.4), we see that this would be the case, so long as

$$h_{\alpha(i)\alpha(j)\alpha(k)} = h_{ijk} \tag{2.8}$$

(where we have suppressed the f_{ij} labels for brevity). To prove this, we have to turn to the properties of the Clebsch-Gordan decomposition. In a quantum mechanical notation, the Clebsch-Gordan coefficient $Y^\alpha_{v_i \overline{v}_j}$ corresponding to the decomposition $\mathbf{3} \otimes \mathbf{r}_i \to \mathbf{r}_j$ may be written as the matrix element

$$Y^\beta_{v_i \overline{v}_j} = \langle \mathbf{3}, \mathbf{r}_i; \beta, v_i | \mathbf{3}, \mathbf{r}_i; \mathbf{r}_j, v_j \rangle. \tag{2.9}$$

At the same time, the Clebsch-Gordan coefficients for the (rather trivial) multiplication by the one-dimensional representation $\mathbf{1}_\alpha$ given in (2.5) are given by a 3×3 unitary matrix

$$U_{v_i \overline{v}_{\alpha(i)}} = \langle \mathbf{1}_\alpha, \mathbf{r}_i; 1, v_i | \mathbf{1}_\alpha, \mathbf{r}_i; \mathbf{r}_{\alpha(i)}, v_{\alpha(i)} \rangle. \tag{2.10}$$

(Unitarity can be seen because we take $\mathbf{1}_\alpha \otimes \bar{\mathbf{1}}_\alpha = \mathbf{1}_0$ without any phases, where $\bar{\mathbf{1}}_\alpha$ is the complex conjugate of $\mathbf{1}_\alpha$, and $\mathbf{1}_0$ is the trivial representation.) Inserting a complete set of states, we then have

$$\begin{aligned} Y^\beta_{v_{\alpha(i)} \overline{v}_{\alpha(j)}} &= \langle \mathbf{3}, \mathbf{r}_{\alpha(i)}; \beta, v_{\alpha(i)} | \mathbf{3}, \mathbf{r}_{\alpha(i)}; \mathbf{r}_{\alpha(j)}, v_{\alpha(j)} \rangle \\ &= \langle \mathbf{1}_\alpha, \mathbf{r}_i; \mathbf{r}_{\alpha(i)}, v_{\alpha(i)} | \mathbf{1}_\alpha, \mathbf{r}_i; 1, v_i \rangle \langle \mathbf{3}, \mathbf{r}_i; \beta, v_i | \mathbf{3}, \mathbf{r}_i; \mathbf{r}_j, v_j \rangle \\ &\qquad \times \langle \mathbf{1}_\alpha, \mathbf{r}_j; 1, v_j | \mathbf{1}_\alpha, \mathbf{r}_j; \mathbf{r}_{\alpha(j)}, v_{\alpha(j)} \rangle \\ &= U^*_{v_i \overline{v}_{\alpha(i)}} Y^\beta_{v_i \overline{v}_j} U_{v_j \overline{v}_{\alpha(j)}}. \end{aligned} \tag{2.11}$$

Since $U_{v_i \overline{v}_{\alpha(i)}}$ is unitary, the superpotential relation (2.8) then follows directly from the definition (2.4). Thus we have now shown that permutations generated by the map (2.5) are indeed good symmetries of the quiver gauge theory.

Finally, before turning to the rephasing symmetries, we make the key observation that the $\mathcal{A}$ type permutation symmetries do indeed form a group. The reason for this is simply that the one dimensional representations of a finite group are all of the representations of the Abelianization of that group, and the Kronecker product acts as group multiplication for these representations. We therefore conclude that the group formed by $\mathcal{A}$ type permutations is given by

$$\{\mathcal{A}\} = \bar{\Gamma} \equiv \Gamma / [\Gamma, \Gamma]. \tag{2.12}$$

2.2.2 Rephasing symmetries

The second class of symmetries that may arise in the quiver theory are discrete (global) U(1) transformations acting on the nodes of the quiver. As discussed in [5,6], these rephasing symmetries may be constructed by assigning discrete U(1) charges q_i to node i. The SU($n_i N$) adjoint gauge multiplets are of course inert under this transformation. However, the bifundamentals $\Phi^{ij}_{f_{ij}}$ stretching from the ith node to the jth node pick up a phase

$$\Phi^{ij}_{f_{ij}} \to \omega^{q_i - q_j} \Phi^{ij}_{f_{ij}}, \tag{2.13}$$

where ω is a primitive k-th root of unity (to be determined below from the anomaly cancellation conditions).

To ensure that the above transformation is a symmetry of the quiver theory, we must demand that it both leaves the superpotential invariant and that it is anomaly free. Invariance of the superpotential is of course automatically satisfied, so only the anomaly condition comes in to restrict the discrete charges. As demonstrated in [6], vanishing of the chiral anomaly at the i-th node demands that

$$\omega^{\sum_j a_{ij}\tilde{q}_j} = 1, \tag{2.14}$$

where we recall that a_{ij} is the adjacency matrix, and $\tilde{q}_i \equiv q_i n_i N$ is the charge weighted by the rank of the i-th gauge group. Solutions to the above fall into two classes. The first class consists of continuous global U(1) symmetries, which happens whenever $\tilde{q}_i$ is a zero eigenvector of the adjacency matrix. The second class occurs whenever the components of the vector $\sum_j a_{ij}\tilde{q}_j$ share a common divisor k (which can be taken to be integral by appropriate scaling of the charges). In other words [6]

$$\sum_j a_{ij}\tilde{q}_j \equiv 0 \pmod{k} \qquad \text{for } k \in \mathbb{Z}. \tag{2.15}$$

Given a set of charges $\tilde{q}_i$ solving the above equation, the chiral multiplets are then rephased according to

$$\Phi^{ij}_{f_{ij}} \to e^{\frac{2\pi i}{kN}\left(\frac{\tilde{q}_i}{n_i} - \frac{\tilde{q}_j}{n_j}\right)} \Phi^{ij}_{f_{ij}}. \tag{2.16}$$

Note, however, that since a transformation in the center of $\mathrm{SU}(n_i N)$ at node i may be written as $\Phi^{ij}_{f_{ij}} \to e^{2\pi i/n_i N}\Phi^{ij}_{f_{ij}}$, the weighted charges $\tilde{q}_i$ are only well defined mod k. In addition, this demonstrates that such a rephasing symmetry is an order k element.

While (2.15) provides the only condition on the rephasing symmetries, it does not provide a constructive procedure for obtaining the $\mathcal{B}$ type (non-central) and $\mathcal{C}$ type (central) transformations. Nevertheless, in practice, it is not too difficult to search for and obtain a consistent set of charge assignments yielding the appropriate discrete transformations. This will be demonstrated below in the examples. For now, we simply note that, for an $\mathcal{A}$ type transformation generated by $\mathbf{1}_\alpha$ and a $\mathcal{B}$ type transformation specified by the charges $\tilde{q}_i$, we may evaluate their commutator expression

$$\mathcal{A}^{-1}\mathcal{B}^{-1}\mathcal{A}\mathcal{B}: \quad \Phi^{ij}_{f_{ij}} \to e^{\frac{2\pi i}{kN}\left(\frac{\tilde{q}_i - \tilde{q}_{\alpha(i)}}{n_i} - \frac{\tilde{q}_j - \tilde{q}_{\alpha(j)}}{n_j}\right)} \Phi^{ij}_{f_{ij}}. \tag{2.17}$$

Identifying this with a central element $\mathcal{C}$ (*i.e.* $\mathcal{AB} = \mathcal{BAC}$) shows that the U(1) charges $\tilde{s}_i$ corresponding to $\mathcal{C}$ are given by

$$\tilde{s}_i = \tilde{q}_i - \tilde{q}_{\alpha(i)} \pmod{k}. \tag{2.18}$$

Evaluating the commutator of $\mathcal{A}$ and $\mathcal{C}$ then gives rise to a potentially new symmetry element $\mathcal{D}$ with U(1) charges given by

$$\begin{aligned} \tilde{t}_i &= \tilde{s}_i - \tilde{s}_{\alpha(i)} \pmod{k} \\ &= \tilde{q}_i - 2\tilde{q}_{\alpha(i)} + \tilde{q}_{\alpha(\alpha(i))}. \end{aligned} \tag{2.19}$$

In order for $\mathcal{C}$ to be central, we demand that $\mathcal{D}$ is gauge equivalent to the identity. Perhaps the simplest way for this to occur is for $\tilde{t}_i$ to vanish at each node. This gives rise to a sufficient condition for $\mathcal{C}$ to be central

$$2\tilde{q}_{\alpha(i)} = \tilde{q}_i + \tilde{q}_{\alpha(\alpha(i))} \pmod{k}. \tag{2.20}$$

This condition, along with the anomaly requirement (2.15), may be used as a guide for constructing the appropriate $\mathcal{B}$ type rephasing symmetries of the quiver. We note, however, that while (2.20) is a sufficient condition, there are other possibilities that make $\mathcal{C}$ central as well. One case that often shows up is when the nodes are all of the same rank. In this case, instead of demanding the vanishing of $\tilde{t}_i$, it is sufficient to ensure that they have a common value

$$\tilde{t}_i = \tilde{t}_j \pmod{k} \qquad \text{when all } n_i = n_j. \tag{2.21}$$

This case arises in particular for abelian orbifolds of the form $\mathbb{C}^3/\mathbb{Z}_n$.

Note that the charge condition (2.20) may be iterated to yield a general solution for the charges

$$\tilde{q}_{\alpha^n(i)} = n\tilde{q}_{\alpha(i)} - (n-1)\tilde{q}_i \pmod{k}, \qquad n = 1, 2, \ldots, \tag{2.22}$$

in terms of only two charges $\tilde{q}_i$ and $\tilde{q}_{\alpha(i)}$. Recalling that this rephasing is an order k operation, we may set $n = k$ in the above expression to obtain $k(\tilde{q}_{\alpha(i)} - \tilde{q}_i) = 0 \pmod{k}$, which is clearly always satisfied for integer charges $\tilde{q}_i$. This ensures the consistency of the charge requirement (2.20).

2.3 The Heisenberg group

As indicated in (2.12), the $\mathcal{A}$ type permutation symmetries close to form a group isomorphic to $\bar{\Gamma}$, the Abelianization of the orbifold action Γ. Ideally, we would be able to demonstrate explicitly that the $\mathcal{B}$ type rephasing symmetries would also form a group isomorphic to $\bar{\Gamma}$. However, we have as yet been unable to show this in a general manner. Nevertheless, in practice, and as indicated in the examples, given the proper identification of the $\mathcal{A}$ symmetries, it is straightforward to construct the appropriate $\mathcal{B}$ generators consistent with (2.15) and either (2.20) or (2.21). (The $\mathcal{C}$ generators then follow as a direct consequence of commuting $\mathcal{A}$ and $\mathcal{B}$.)

The construction of the $\mathcal{B}$ generators is guided by noting that since $\{\mathcal{A}\}$ is Abelian, it necessarily decomposes into a set of cyclic groups

$$\{\mathcal{A}\} = \bar{\Gamma} = \mathbb{Z}_{a_1} \otimes Z_{a_2} \otimes \cdots . \tag{2.23}$$

We may then focus on a single $\mathbb{Z}_{a_i}$ group at a time. Being Abelian, this group may be generated by a single element (*i.e.* some particular $\mathbf{1}_\alpha$) which we denote by $\mathcal{A}_i$, and which cycles the nodes of the quiver. The set of nodes of the quiver then fall into distinct orbits of $\mathcal{A}_i$. In general, the rephasing generator $\mathcal{B}_i$ corresponding to this $\mathcal{A}_i$ may be obtained by the linear charge assignment given in (2.22), while in some cases iteration of (2.21) may be required. In any case, the related central element $\mathcal{C}_i$ is given by the charge assignment of (2.18).

In this way, the complete group of global symmetries of the quiver is constructed as a direct product of individual Heisenberg groups generated by the elements $\mathcal{A}_i\mathcal{B}_i = \mathcal{B}_i\mathcal{A}_i\mathcal{C}_i$ (where i labels the group, and is not summed over). In other words, given the decomposition (2.23), the discrete symmetry group takes the form

$$\mathrm{Heis}(\mathbb{Z}_{a_1}, \mathbb{Z}_{a_1}) \otimes \mathrm{Heis}(\mathbb{Z}_{a_2}, \mathbb{Z}_{a_2}) \otimes \cdots , \tag{2.24}$$

which is just the decomposition (1.5) highlighted in the introduction.

3 String theory interpretation and torsion cycles

The result (2.24) takes on added physical significance when the quiver gauge theory is related to the dual string picture. The general framework is of course clear: following the general ideas of [1] and [4], we wish to identify the symmetry generators $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ in the field theory with corresponding operators counting the number of wrapped F-strings, D-strings and D3-branes, respectively, in the dual string theory. Some subtleties arise, however, in making precise the field theory/string theory connection in cases were Γ is non-Abelian. As a result, we find it worthwhile to make a distinction between:

1. The orbifold quiver gauge theory.
2. The near horizon manifold S^5/Γ.
3. The string theory orbifold $\mathbb{C}^3/\Gamma$.

In the first case, we have demonstrated that the orbifold quiver gauge theory admits the finite Heisenberg group (1.5) as a group of global symmetries of the field theory. Based on AdS/CFT, this field theory ought to be dual to string theory on the horizon manifold S^5/Γ. To show that the symmetries match on both sides of the duality, we need information on the homology classes of S^5/Γ. After all, we expect the $\mathcal{A}$ transformations to be identified with operators counting the number of F-strings in the dual string theory. The structure of eigenvalues of these operators is clearly determined by the first homology class $H_1(S^5/\Gamma)$. Furthermore, based on S-duality, the $\mathcal{B}$ transformations may be associated with operators counting the number of D-strings; these operators are also valued in $H_1(S^5/\Gamma)$. Finally, the $\mathcal{C}$ transformations match the operators counting wrapped D3-branes and is determined by the third homology class, $H_3(S^5/\Gamma)$. While S^5 itself has no non-trivial cycles, orbifolds of S^5 may admit torsion one and three-cycles, which are exactly what is required to allow this duality to work.

There is a slight subtlety in the identification of the operators of $\mathcal{C}$. Namely, it may seem curious that the D3-brane number operator is somehow related to the F-string and D-string number operators. However, we note that Poincaré duality relates the torsion free part of homology groups by $(H_p)_{TF} = (H_{(d-p)})_{TF}$, while the (cyclic) torsion parts of the homology are related by $\mathrm{Tors}(H_p) = \mathrm{Tors}(H_{(d-p-1)})$. We are in particular interested in the torsion parts; on the five sphere, $\mathrm{Tors}(H_1) = \mathrm{Tors}(H_{(5-1-1)}) = \mathrm{Tors}(H_{(3)})$ (see for example [13]). Therefore, the torsion cycles that F-strings or D-strings may wrap are isomorphic to torsion cycles that D3-branes may wrap. As was shown explicitly in [9], in the more general case of branes placed in generic toric singularities, this is no longer the case, and one can have that $H_1(H)$ is not isomorphic to $H_3(H)$ where H is the near horizon space.

For S^5/Γ, on the other hand, the non-trivial homology is given by (1.3). As a result, the dual picture of F-strings and D-strings wrapping torsion cycles gives rise to the identical Heisenberg group (1.5) that was obtained in the field theory analysis. This statement is the extension of [1, 4] to the non-Abelian case.

Although the duality between the symmetries of the quiver gauge theory and the near horizon manifold S^5/Γ is clear, the connection with string theory on $\mathbb{C}^3/\Gamma$ is less so (at least in the non-Abelian case). This is because a rigorous understanding of D3-branes near a non-Abelian orbifold singularity involves the generalization of [10–12] to the non-Abelian case, and this is as yet incomplete. Here we simply make an observation on the structure of twisted sectors in string theory on orbifolds. In string theory we are mainly familiar with global orbifolds by Abelian groups such as $\mathbb{Z}_N$ or $\mathbb{Z}_N \times \mathbb{Z}_N$. In this case, the space of twisted sectors is in correspondence with elements of the group excluding the identity element. In general, however, twisted sectors are in correspondence with conjugacy classes [14]. Consider a string field X whose boundary conditions are twisted by an element $g \in \Gamma$, namely $X(\sigma + 2\pi) = gX(\sigma)$. For any element $h \in \Gamma$ we can act on the previous relation on the left: $hX(\sigma + 2\pi) = hgX(\sigma) = (hgh^{-1})\, hX(\sigma)$. This means that strings twisted by hgh^{-1} are all in the same sector. Thus twisted sectors are defined only up to conjugacy classes. That is, there is one sector per conjugacy class in Γ.

Since the field theory states are classified by the Abelianization of Γ, to make a connection between the quiver and orbifold pictures, we must appropriately relate the conjugacy classes of Γ with the Abelianization of Γ. In the non-Abelian case, however, this relation is not so clear. Moreover, unlike the Abelianization of a group, which itself is a group, there is no natural group structure on conjugacy classes. Nevertheless, we fully expect that (1.5) properly describes the global symmetries of the theory on both the gauge theory and the string theory sides of the duality.

4 Examples

As indicated above, we are interested in $\mathcal{N} = 1$ quiver theories that may be obtained by orbifolding $\mathcal{N} = 4$ super-Yang Mills by a group Γ. To ensure $\mathcal{N} = 1$ supersymmetry, Γ must be restricted to be a discrete subgroup of SU(3). In fact, all such discrete subgroups have been classified [15], and many of the resulting quiver gauge theories have been described in [16–18].

We follow the discussion of [16] concerning the non-Abelian discrete subgroups of SU(3). The relevant subgroups which are not contained in SU(2) fall into two infinite series, $\Delta(3n^2)$ and $\Delta(6n^2)$, as well as the exceptional subgroups $\Sigma(36)$, $\Sigma(60)$, $\Sigma(72)$, $\Sigma(168)$, $\Sigma(216)$, $\Sigma(360)$ and $\Sigma(36 \times 3)$, $\Sigma(60 \times 3)$, $\Sigma(168 \times 3)$, $\Sigma(216 \times 3)$, $\Sigma(360 \times 3)$. We note that the subgroups in the infinite series are subgroups of SU(3) for $n = 0 \mod 3$, and $\mathrm{SU}(3)/\mathbb{Z}_3$ for $n \neq 0 \mod 3$. Similarly, the latter set of exceptional subgroups are subgroups of SU(3), while the former set are subgroups of $\mathrm{SU}(3)/\mathbb{Z}_3$ [16].

In Appendix A, we compute the Abelianization of several of these groups. The results are given in Table 1. Note that the Abelianization of Γ is not necessarily related to the center of SU(3), as groups such as $\mathbb{Z}_2$ and $\mathbb{Z}_4$ may arise for $\bar{\Gamma}$. In most cases, $\bar{\Gamma}$ is given by a single cyclic group $\mathbb{Z}_k$, in which case the discrete symmetry group of the quiver is a single copy of $\mathrm{Heis}(\mathbb{Z}_k \times \mathbb{Z}_k)$. The case of $\Delta(3n^2)$ for $n = 0 \mod 3$ is somewhat interesting, though, as its Abelianization contains two cyclic factors. We will highlight this case below by considering $\Delta(27)$. However, we start with the familiar example of the $\mathbb{Z}_3$ orbifold in order to introduce the language of discrete symmetries constructed from one-dimensional representations of Γ.

Γ	$\bar{\Gamma}$	
$\Delta(3n^2)$	$\mathbb{Z}_3 \times \mathbb{Z}_3$	for $n = 0 \mod 3$
	$\mathbb{Z}_3$	for $n \neq 0 \mod 3$
$\Delta(6n^2)$	$\mathbb{Z}_2$	
$\Sigma(36)$	$\mathbb{Z}_4$	
$\Sigma(36 \times 3)$	$\mathbb{Z}_4$	

Table 1: Some subgroups Γ of SU(3) and their Abelianization $\bar{\Gamma}$.

4.1 The $\mathbb{Z}_3$ quiver

The discrete symmetries of the $\mathbb{Z}_3$ quiver formed the basis of the analysis of Gukov, Rangamani and Witten in [1]. Although this group is Abelian, we find this example instructive as it enables us to emphasize the rôle of one-dimensional representations as generators of the $\mathcal{A}$ type permutation symmetries.

The cyclic group $\mathbb{Z}_3$ has only one generator, which we call A, and which satisfies

$$A^3 = 1. \tag{4.1}$$

Since $\mathbb{Z}_3$ is Abelian, it only has one-dimensional representations. For the same reason, the conjugacy classes are given by just the individual group elements. The character table is then

$$\begin{array}{c||c|c|c} & 1 & A & A^2 \\ \hline\hline \mathbf{1}_0 & 1 & 1 & 1 \\ \hline \mathbf{1}_1 & 1 & \gamma & \gamma^2 \\ \hline \mathbf{1}_2 & 1 & \gamma^2 & \gamma \end{array} \tag{4.2}$$

where γ is a cube root of unity.

To construct the $\mathbb{Z}_3$ quiver, we must specify how it acts on the space $\mathbb{C}^3$. This corresponds to specifying an appropriate faithful three dimensional representation to act on the global index, which is also required to be a subgroup of SU(3). We take this to be

$$A = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix}. \tag{4.3}$$

Now it is easy to see what happens. The three dimensional representation is a reducible representation: $\mathbf{3} = \mathbf{1}_1 \oplus \mathbf{1}_1 \oplus \mathbf{1}_1$. It is three copies of the same representation, and so in the quiver we expect three arrows all pointing in the same direction from a given node. Also, we expect a global SU(3) symmetry because one may recombine

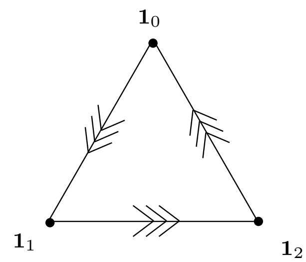

Figure 1: The $\mathbb{Z}_3$ quiver.

like representations into each other. We will check this at the end of the calculation[2]. The structure of the quiver is given by

$$\mathbf{3} \otimes \mathbf{1}_i = \mathbf{1}_{i+1} + \mathbf{1}_{i+1} + \mathbf{1}_{i+1}. \tag{4.4}$$

Thus the quiver, shown in Fig. 1, has three arrows pointing from one node to the next in the cyclic order $\mathbf{1}_0 \to \mathbf{1}_1 \to \mathbf{1}_2$. We label the bifundamental fields by $C^i_{a,b}$ where a (b) labels the node that the arrow points from (to), and i is a global SU(3) index labeling which arrow of the three that is being referred to, and which directly corresponds to the global index in the $\mathcal{N} = 4$ theory from which the above orbifold theory descends.

To compute the superpotential, we note that the Clebsch-Gordan coefficients are all trivial (taken to be unity in our conventions), as there is only one way of combining one-dimensional representations. In this case, the superpotential is easy to deduce, and is simply $W = \epsilon_{ijk} C^i_{01} C^j_{12} C^k_{20}$. As claimed, there is a global SU(3) symmetry which directly corresponds to mixing the three irreducible representations in the $\mathbf{3}$ (reducible) representation.

Given the quiver and the superpotential, we are now in a position to highlight the global symmetries $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$. Starting with the $\mathcal{A}$ type shift symmetry, we recall that it is generated by the action of the one-dimensional representations on the nodes and links of the quiver according to (2.5). Taking $\mathbf{1}_1$ to generate the $\mathcal{A}$ symmetry,

[2] The global SU(3) symmetry is a statement that there are three identical representations acting on the global index, and this remains as a global symmetry. This is analogous to the unbroken gauge factors. Recall that in the regular representation, an n-dimensional representation appears n times, and it is this fact that leads to the unbroken U(n) gauge group. In the holographic limit, such symmetries become SU(nN).

we see that it simply maps the nodes cyclicly $\mathbf{1}_0 \to \mathbf{1}_1 \to \mathbf{1}_2$. The mapping of the fields is also easy, as all Clebsch-Gordan coefficients are trivial, and is given simply by

$$\mathcal{A}: \qquad C^i_{a,b} \to C^i_{a+1,b+1} \tag{4.5}$$

(where the node labels are taken mod 3). This is clearly a symmetry of the superpotential.

To obtain the $\mathcal{B}$ and $\mathcal{C}$ rephasing symmetries, we start with the adjacency matrix

$$\mathbf{a} = \begin{pmatrix} 0 & 3 & -3 \\ -3 & 0 & 3 \\ 3 & -3 & 0 \end{pmatrix}. \tag{4.6}$$

The anomaly equation (2.15) takes the form

$$\mathbf{a} \cdot \mathbf{v} \equiv 0 \pmod 3, \tag{4.7}$$

and can be satisfied by any vector $\mathbf{v}$ of integers. Since all gauge groups have the same rank, an appropriate choice of charge vector satisfying (2.21) with $\tilde{t}_i = 1 \mod 3$ is given by $\mathbf{v} = (0, 0, 1)$. On the fields, this corresponds to a rephasing symmetry

$$\begin{aligned} \mathcal{B}: \qquad & C^i_{0,1} \to C^i_{0,1}, \\ & C^i_{1,2} \to \omega^{-1} C^i_{1,2}, \\ & C^i_{2,0} \to \omega^1 C^i_{1,2}, \end{aligned} \tag{4.8}$$

where $\omega = \exp(2\pi i/(3N))$. Note that this symmetry applied three times gives a member of the center of the gauge group, and so is gauge equivalent to the identity.

Clearly $\mathcal{A}$ and $\mathcal{B}$ do not commute. They in fact close on the rephasing symmetry given by the vector $\mathbf{v}' = (1, 0, -1)$, acting on the fields as

$$\begin{aligned} \mathcal{C}: \qquad & C^i_{0,1} \to \omega C^i_{0,1}, \\ & C^i_{1,2} \to \omega C^i_{1,2}, \\ & C^i_{2,0} \to \omega^{-2} C^i_{1,2}. \end{aligned} \tag{4.9}$$

Although it is not obvious here, $\mathcal{A}$ and $\mathcal{C}$ commute (up to a gauge transformation). This is because the charges needed to close the $\mathcal{AC}$ commutator are $\mathbf{v}'' = (-2, 1, 1)$, which rephases in the exact same way as $\mathbf{v}'' = (-3, 0, 0)$. The latter is in the center of the gauge group because all charges are divisible by three. Examination of (4.5), (4.8) and (4.9) indicates that the symmetry generators obey the relations

$$\begin{aligned} &\mathcal{A}^3 = \mathcal{B}^3 = \mathcal{C}^3 = 1, \quad \mathcal{AB} = \mathcal{BAC}, \\ &\mathcal{AC} = \mathcal{CA}, \quad \mathcal{BC} = \mathcal{CB}, \end{aligned} \tag{4.10}$$

up to the center of the gauge group [1]. As expected, this is just the finite Heisenberg group $\mathrm{Heis}(\mathbb{Z}_3 \times \mathbb{Z}_3)$.

4.2 The $\Delta(27)$ quiver

This particular orbifold has been much studied in the literature, and is one of the simplest examples of a non-Abelian orbifold [16, 19–21, 17, 18, 22]. Here we will try to be as detailed as possible, so that the general structure of the discrete symmetries becomes apparent in this example.

The group theory details for $\Delta(27)$ are given in Appendix B. Here we note that, for any $n = 0 \mod 3$, the group $\Delta(3n^2)$ is a subgroup of the full SU(3), and has $n^2/3 - 1$ three-dimensional representations and 9 one-dimensional representations. The three-dimensional representations may be labeled by integer pairs i, j with $0 \leq i, j \leq n$, along with the equivalence $\mathbf{3}_{i,j} = \mathbf{3}_{j,-i-j} = \mathbf{3}_{-i-j,i}$ (where labels are taken mod n). The $\mathbf{3}_{0,0}$, $\mathbf{3}_{n/3,n/3}$ and $\mathbf{3}_{2n/3,2n/3}$ representations are reducible, and fall apart into the nine one-dimensional representations, which we may label as $\mathbf{1}_{i,j}$ as follows:

$$\mathbf{3}_{i\times n/3, i\times n/3} \quad \rightarrow \quad \mathbf{1}_{i,0}, \mathbf{1}_{i,1}, \mathbf{1}_{i,2}. \tag{4.11}$$

In the present case of $n = 3$, we only have two three-dimensional representations (which we therefore label $\mathbf{3}$ and $\bar{\mathbf{3}}$). Using the $\mathbf{3}$ representation as the faithful representation that acts on the global symmetry index, one may easily deduce the structure of the quiver (see the table of tensor products in Appendix B). This is given in Fig. 2. We label the fields pointing from the $\mathbf{1}_{i,j}$ node as $A_{i,j}$ and those pointing to the $\mathbf{1}_{i,j}$ as $B_{i,j}$. In addition, we label the three other fields (pointing from the $\mathbf{3}$ to the $\bar{\mathbf{3}}$) as C_i, $i = 0, 1, 2$.

The superpotential may be obtained using the Clebsch-Gordan coefficients given in Appendix B. The result is

$$W = \alpha_{i,j,k} B_{i,j} A_{i,j} C_k, \tag{4.12}$$

with the coefficients $\alpha_{i,j,k}$ given in Table 2. More succinctly, we have

$$\alpha_{i,j,k} = \delta_{i-1,k}\gamma^{-j} - \delta_{i+1,k}, \tag{4.13}$$

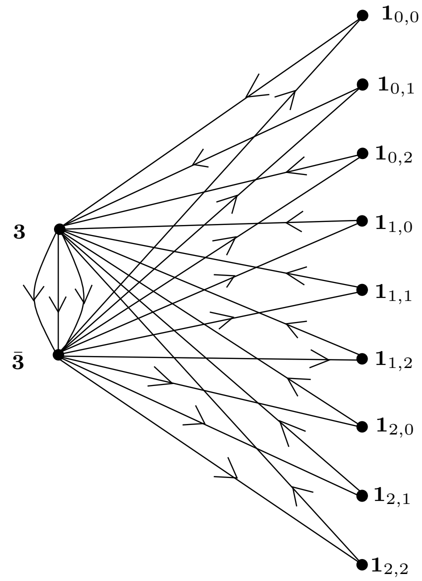

Figure 2: The $\Delta(27)$ quiver.

where all indices are to be taken modulo 3, and where $\gamma^3 = 1$.

To construct the shift symmetries, we may begin with the $\mathbb{Z}_3$ symmetry generated by $\mathbf{1}_{1,0}$ which takes $\mathbf{1}_{i,j} \to \mathbf{1}_{i+1,j}$ as well as $\mathbf{3} \to \mathbf{3}$ and $\bar{\mathbf{3}} \to \bar{\mathbf{3}}$. Although the $\mathbf{3}$ and $\bar{\mathbf{3}}$ nodes are inert under this transformation, the C_i links are permuted. The action of this symmetry , which we denote $\mathcal{A}$, on the bifundamentals is thus

$$\mathcal{A}: \quad A_{i,j} \to A_{i+1,j}, \qquad B_{i,j} \to B_{i+1,j}, \qquad C_k \to C_{k+1}. \tag{4.14}$$

The Abelianization of $\Delta(27)$ contains a second $\mathbb{Z}_3$ factor, and so we expect a second $\mathbb{Z}_3$ symmetry of the quiver. Noting that the nine one-dimensional representations form the group $\mathbb{Z}_3 \times \mathbb{Z}_3$ under ordinary multiplication ($\mathbf{1}_{i,j} \times \mathbf{1}_{k,l} = \mathbf{1}_{i+k,j+l}$), we may take this second $\mathbb{Z}_3$ to be generated by the action of $\mathbf{1}_{0,1}$. This acts on the second index of the $B_{i,j}$ and $A_{i,j}$ fields. However, this is accompanied by additional rephasings as well. The action of this $\mathbb{Z}_3'$, which we call $\mathcal{A}'$, is[3]

$$\mathcal{A}': \quad A_{i,j} \to A_{i,j+1}, \qquad B_{i,j} \to \gamma^i B_{i,j+1}, \qquad C_k \to \gamma^{1-k} C_k. \tag{4.15}$$

[3]Equivalently, one may choose to rephase $A_{i,j}$ and leave $B_{i,j}$ alone. This symmetry is pure gauge, given by simultaneously using the center of the gauge groups $\mathbf{3}$ and $\bar{\mathbf{3}}$, hence leaving the C fields unchanged.

$\alpha_{i,j,k}$	C_0	C_1	C_2
$B_{0,j}A_{0,j}$	0	-1	1
	0	-1	γ^2
	0	-1	γ
$B_{1,j}A_{1,j}$	1	0	-1
	γ^2	0	-1
	γ	0	-1
$B_{2,j}A_{2,j}$	-1	1	0
	-1	γ^2	0
	-1	γ	0

Table 2: The superpotential coefficients $\alpha_{i,j,k}$ multiplying $B_{i,j}A_{i,j}C_k$. Here $\gamma = e^{2\pi i/3}$ is a cube root of unity.

We note that $\mathbb{Z}_3$ and $\mathbb{Z}_3'$ in (4.14) and (4.15) do not strictly commute. However, they commute up to a rephasing of the fields

$$A_{i,j} \to A_{i,j}, \qquad B_{i,j} \to \gamma B_{i,j}, \qquad C_k \to \gamma^2 C_k, \tag{4.16}$$

which is in the center of the gauge group associated with the $\bar{\mathbf{3}}$ representation. This is because γ is a third root of unity, and so is also a $3N$-th root of unity, thus ensuring that the above rephasing is in the center of the SU$(3N)$ gauge group associated with the $\bar{\mathbf{3}}$ node. Thus the $\Delta(27)$ quiver does in fact admit a $\mathbb{Z}_3 \times \mathbb{Z}_3$ symmetry. Incidentally, we note that the quiver diagram shown in Fig. 2 actually admits an S_9 symmetry permuting the nine 'singlet' nodes. The superpotential, however, is only invariant under the $\mathbb{Z}_3 \times \mathbb{Z}_3$ subgroup of the full S_9 permutation group.

We now turn to the rephasing symmetries. Since the $\mathcal{A}$ type symmetries generate $\mathbb{Z}_3 \times \mathbb{Z}_3$, we expect the $\mathcal{B}$ type rephasings to generate this identical group. These rephasings are relatively easy to deduce from the results for the $\mathbb{Z}_3$ orbifold theory [1] given above. There, the appropriate rephasings were given by the charges $(0, 0, 1)$ on the three nodes of the quiver. Therefore, by extension, we find the charge assignments for the nine SU(N) nodes to be correctly given by $(0, 0, 0, 0, 0, 0, 1, 1, 1)$ for what we call $\mathcal{B}$, and $(0, 0, 1, 0, 0, 1, 0, 0, 1)$ for what we call $\mathcal{B}'$ (the SU$(3N)$ nodes are assigned zero charge). This generates the rephasings

$$\mathcal{B}: \qquad A_{i,j} \to \omega A_{i,j}, \qquad B_{i,j} \to \omega^{-1} B_{i,j}, \quad i = 2 \text{ only}, \tag{4.17}$$

$$\mathcal{B}': \qquad A_{i,j} \to \omega A_{i,j}, \qquad B_{i,j} \to \omega^{-1} B_{i,j}, \quad j = 2 \text{ only}, \tag{4.18}$$

where the other fields are not rephased. Here $\omega = \exp(2\pi i/(3N))$ is a primitive $3N$-th root of unity.

It is clear that the primed symmetries commute with unprimed symmetries. However, $\mathcal{A}$ and $\mathcal{B}$ do not commute, and $\mathcal{A}'$ and $\mathcal{B}'$ do not commute. Instead, they commute up to the respective symmetries

$$\mathcal{C}: \qquad A_{i,j} \to \omega^{i-1} A_{i,j}, \quad B_{i,j} \to \omega^{-(i-1)} B_{i,j}, \tag{4.19}$$

$$\mathcal{C}': \qquad A_{i,j} \to \omega^{j-1} A_{i,j}, \quad B_{i,j} \to \omega^{-(j-1)} B_{i,j}. \tag{4.20}$$

Again, it is clear that primed and unprimed symmetries commute. It is also clear that $\mathcal{B}$ and $\mathcal{C}$ (primed or not) commute. Finally, $\mathcal{A}$ and $\mathcal{C}$ also commute, but only up to the center of the gauge group. In particular, the member of the center of the gauge group that closes the $\mathcal{AC}$ commutator is

$$\begin{aligned} &A_{i,j} \to \omega^{-2} A_{i,j}, \quad B_{i,j} \to \omega^{2} B_{i,j}, \quad i = 0, \\ &A_{i,j} \to \omega A_{i,j}, \quad B_{i,j} \to \omega^{-1} B_{i,j}, \quad i \neq 0, \end{aligned} \tag{4.21}$$

and the member of the center of the gauge group that closes the $\mathcal{A}'\mathcal{C}'$ commutator is

$$\begin{aligned} &A_{i,j} \to \omega^{-2} A_{i,j}, \quad B_{i,j} \to \omega^{2} B_{i,j}, \quad j = 0, \\ &A_{i,j} \to \omega A_{i,j}, \quad B_{i,j} \to \omega^{-1} B_{i,j}, \quad j \neq 0, \end{aligned} \tag{4.22}$$

where in both cases the C_k are unchanged. These two rephasings can be seen to correspond to the center of the gauge groups by assigning charge -3 to each of the SU$(3N)$ nodes and then assigning $(3,3,3,0,0,0,0,0,0)$ and $(3,0,0,3,0,0,3,0,0)$ charge vectors to the SU(N) nodes, respectively.

Thus the structure of the global symmetries is that each $\mathbb{Z}_3$ shift symmetry ($\mathcal{A}$ or $\mathcal{A}'$) associated with a one-dimensional representation has a corresponding rephasing $\mathbb{Z}_3$ symmetry ($\mathcal{B}$ or $\mathcal{B}'$). These two symmetries close on a final $\mathbb{Z}_3$ symmetry ($\mathcal{C}$ or $\mathcal{C}'$). All generators are of order three (up to the center of the gauge group), and the primed and unprimed ones commute. As a result, taken together, they form a direct product of two Heisenberg groups

$$\text{Heis}(\mathbb{Z}_3 \times \mathbb{Z}_3) \times \text{Heis}(\mathbb{Z}_3 \times \mathbb{Z}_3), \tag{4.23}$$

in agreement with the expectation from (1.5), where we note that the Abelianization of $\Delta(27)$ is just $\mathbb{Z}_3 \times \mathbb{Z}_3$.

5 Conclusions

Our work demonstrates that quiver gauge theories obtained as worldvolume theories on a stack of N D3-branes on the singular point of $\mathbb{C}^3/\Gamma$ where Γ is non-Abelian admit a discrete group of global symmetries which may be expressed as a product of Heisenberg groups. It is worth mentioning, however, that the actual matching which we perform is with string theory on the near horizon manifold $\mathrm{AdS}_5 \times S^5/\Gamma$. This highlights the importance of the decoupling limit which is accompanied with the decoupling of various U(1)'s, some of them anomalous. We believe a similar structure should exist in the general case of quiver gauge theories obtained as worldvolume theories on a stack of N D3-branes placed at the singular point of a toric variety which can be obtained as a non-Abelian orbifold of some other toric variety.

More generally, our series of papers [4–6] suggests that field theoretical methods might help in answering in full generality the question of the spectrum of D-brane charges in string theories on curved backgrounds and with fluxes whenever they admit field theory duals. It has clearly been established that the Heisenberg group structure is present in a variety of situations, including cascading theories and Seiberg dual phases. The larger question towards which our work points is the computation of the spectrum on branes using AdS/CFT, therefore predicting the outcome of the corresponding generalized cohomology theory classifying the D-brane charges in string theory. For example, a question arises motivated by a recent comment made in [23] about twisted K-theory being able to classify universality classes of baryonic vacua in the Klebanov-Strassler background. It is worth point out that the argument of [23] is entirely based on the geometry of fluxes in the supergravity background.

Given our previous work with cascading quiver gauge theories [5], it is natural to suspect that field theoretic methods along the lines described here should be relevant to understanding the spectrum of D-brane charges in those backgrounds. Note that, to a large extent, the states charged under the symmetries $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are determinant operators in the field theory, also referred to as baryonic. Here we would also like to stress that while we have considered certain global symmetries, we have not considered all possible symmetries, and a full classification and their dual interpretation would be enlightening. For example, the continuous $U(1)$ baryon number symmetries (associated with various fractional branes) do not commute with the shift operators, and close on other baryon $U(1)$ symmetries. We hope to return to some of these issues in the future.

Acknowledgments

We are thankful to P. de Medeiros, Y.-H. He, C. Herzog, I. Kriz, D. Morrison and B. Uribe for various comments about mathematical aspects of our work. JTL is especially grateful to S. Teraguchi for raising the issue of non-Abelian orbifolds and for suggesting that the previous work on discrete symmetries may be extended to cover more general orbifold constructions. This work is supported in part by the US Department of Energy under grant DE-FG02-95ER40899, the National Science Foundation under grant PHY99-07949, the Israel Science Foundation, and the German-Israeli Project (DIP) program (H.52). JTL wishes to acknowledge the hospitality of the Department of Physics at National Taiwan University and the Taiwan National Center for Theoretical Sciences where this project was initiated, and the Kavli Institute for Theoretical Physics where portions of this work were completed.

A The Abelianization of G

We have seen that the Abelianization of the orbifold group Γ (which we denote $\bar{\Gamma} \equiv \Gamma/[\Gamma,\Gamma]$) plays a key rôle in understanding the discrete global symmetries of the orbifold quiver gauge theory. In particular, for a quiver constructed as an orbifold of $\mathcal{N} = 4$ super-Yang Mills by Γ via the process in [8], $\bar{\Gamma}$ (which is isomorphic to the group formed by the one-dimensional representations of Γ under ordinary multiplication) generates a set of permutation symmetries mapping fields to fields and gauge groups to gauge groups. We expect this permutation mapping to work for non-supersymmetric orbifolds as well as supersymmetric ones[4]. However here we restrict ourselves to the supersymmetric case. In this case, Γ is a discrete subgroup of SU(3), and the categorization of these subgroups is known [15]. In this Appendix, we compute the Abelianization of Γ for the following examples:

$$\Delta(3n^2), \qquad \Delta(6n^2), \qquad \Sigma(36), \qquad \Sigma(36\times 3). \tag{A.1}$$

[4]In the non-supersymmetric case, one simply writes two different kinds of arrows in the quiver: one type for fermions, and one type for bosons. Since the gauge factors for both the fermions and bosons are mapped in the same way, one still expects the one-dimensional representations to generate symmetries of the resulting theory, again because the regular representation is used on the gauge indices. One can similarly argue that the (non-super) potential that arises in these cases is still preserved by these symmetries. It is interesting to see that this structure holds even for non-supersymmetric orbifolds.

A.1 The case $\Gamma = \Delta(3n^2)$

The group $\Delta(3n^2)$ is generated by

$$A^3 = B^n = C^n = 1, \tag{A.2}$$

and

$$BA = AC^{-1}, \qquad CA = ABC^{-1} \tag{A.3}$$

(with all other generators commuting). As a result, the commutator subgroup is generated by the elements $\{B^{-1}C^{-1}, BC^{-2}\}$. This is equivalent to taking the generators

$$\{BC, C^3\} \quad \text{for} \quad n = 0 \mod 3; \qquad \{BC, C\} \quad \text{for} \quad n \neq 0 \mod 3. \tag{A.4}$$

Since a generic $\Delta(3n^2)$ element can be written as

$$g = A^a B^b C^c = A^a (BC)^b C^{c-b}, \tag{A.5}$$

it is easy to see that the Abelianization of $\Gamma = \Delta(3n^2)$ is

$$\bar{\Gamma} = \begin{cases} \mathbb{Z}_3 \times \mathbb{Z}_3 & \text{if } n = 0 \mod 3, \\ \mathbb{Z}_3 & \text{if } n \neq 0 \mod 3. \end{cases} \tag{A.6}$$

A.2 The case $\Gamma = \Delta(6n^2)$

The group $\Delta(6n^2)$ is generated by

$$A_1^2 = A_2^3 = B^n = C^n, \tag{A.7}$$

with

$$\begin{aligned} &A_2 A_1 = A_1 A_2^{-1}, \qquad BA_1 = A_1 C, \qquad CA_1 = A_1 B, \\ &BA_2 = A_2 C^{-1}, \qquad CA_2 = A_2 BC^{-1}. \end{aligned} \tag{A.8}$$

Note that $\{A_1, A_2\}$ generate the permutation group S_3. Thus $\Delta(6n^2)$ is isomorphic to $(\mathbb{Z}_n \times \mathbb{Z}_n) \ltimes S_3$. A generic element of $\Delta(6n^2)$ may be written as

$$g = A_1^{a_1} A_2^{a_2} B^b C^c, \tag{A.9}$$

and the commutator subgroup is generated by

$$\{A_2, B, C\}. \tag{A.10}$$

As a result, the Abelianization of $\Gamma = \Delta(6n^2)$ is simply the group generated by A_1. Thus

$$\bar{\Gamma} = \mathbb{Z}_2, \tag{A.11}$$

and is independent of any $n \mod 3$ issues.

A.3 The case $\Gamma = \Sigma(36)$

The group $\Sigma(36)$ is generated by

$$V^4 = A^3 = B^3 = 1, \tag{A.12}$$

along with

$$AV = VB, \qquad BV = VA^{-1}. \tag{A.13}$$

Elements of $\Sigma(36)$ may be written as

$$g = V^v A^a B^b. \tag{A.14}$$

Since the commutator subgroup is generated by

$$\{A, B\}, \tag{A.15}$$

the Abelianization of $\Gamma = \Sigma(36)$ is the group generated by V. This gives

$$\bar{\Gamma} = \mathbb{Z}_4. \tag{A.16}$$

A.4 The case $\Gamma = \Sigma(36 \times 3)$

The group $\Sigma(36 \times 3)$ is a central extension of $\Sigma(36)$. It is generated by

$$V^4 = A^3 = B^3 = C^3 = 1, \tag{A.17}$$

with C a central extension, so that

$$AB = BAC, \qquad AV = VB, \qquad BV = VA^{-1}. \tag{A.18}$$

Elements of $\Sigma(36 \times 3)$ may be written as

$$g = V^v A^a B^b C^c. \tag{A.19}$$

Since A and B no longer commute, we ought to be a bit careful in determining the generators of the commutator subgroup. From the above expressions, we see that the commutator subgroup is generated by $\{A^{-1}B, B^{-1}A^{-1}, C\}$. Multiplying the first two generators in order (and using $A^3 = 1$) gives A. Since this is now part of the commutator subgroup, it can be multiplied with $A^{-1}B$ to obtain B. As a result, the commutator subgroup is generated by

$$\{A, B, C\}, \tag{A.20}$$

so that it is in fact isomorphic to the Heisenberg group $\Delta(27)$. Taking a quotient of $\{V, A, B, C\}$ by $\{A, B, C\}$ demonstrates that the Abelianization of $\Gamma = \Sigma(36 \times 3)$ is the group generated by V. Thus

$$\bar{\Gamma} = \mathbb{Z}_4. \tag{A.21}$$

It is not particularly surprising that this is the same result as the Abelianization of $\Sigma(36)$.

B The group theory of $\Delta(27)$

Although the finite subgroups of SU(3) have been well studied [24, 25, 15], we will try to make it accessible to the careful reader by displaying the basic group theoretic properties used in the above calculations. Here, we consider the case where $\Gamma = \Delta(27)$. This group is a single group in the series of groups $\Delta(3n^2)$ contained in SU(3). We take the presentation of $\Delta(27)$ as follows:

$$A^3 = B^3 = C^3 = 1, \quad AB = BAC, \quad AC = CA, \quad BC = CB. \tag{B.1}$$

This group has nine one-dimensional representations and two three-dimensional representations. These are given in Table 3, where we have chosen a convenient set of labels.

Matching the eleven irreducible representations, the group $\Delta(27)$ also has eleven conjugacy classes. This gives rise to the (partial) character table

	1	C	C^2	A	A^2	B	B^2	AB	A^2B	AB^2	A^2B^2
$\mathbf{1}_{0,0}$	1	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}_{0,1}$	1	1	1	γ	γ^2	1	1	γ	γ^2	γ	γ^2
$\mathbf{1}_{1,0}$	1	1	1	1	1	γ	γ^2	γ	γ	γ^2	γ^2
$\mathbf{1}_{1,1}$	1	1	1	γ	γ^2	γ	γ^2	γ^2	1	1	γ
$\mathbf{1}_{2,1}$	1	1	1	γ	γ^2	γ^2	γ	1	γ	γ^2	1
$\mathbf{3}$	3	3γ	$3\gamma^2$	0	0	0	0	0	0	0	0

(B.2)

The remaining characters are given by complex conjugation of the above representations and characters. Explicitly, we take $\gamma \leftrightarrow \gamma^2$ in the characters, and map the labels for the representations by $\mathbf{3} \to \bar{\mathbf{3}}$ and $\mathbf{1}_{i,j} \to \mathbf{1}_{3-i,3-j}$, where these indices are taken mod 3. Also, note that the conjugacy classes for the last eight columns above contain three elements: the element given, as well as its product with C and C^2.

Rep	A	B	C
$\mathbf{1}_{0,0}$	1	1	1
$\mathbf{1}_{0,1}$	γ	1	1
$\mathbf{1}_{0,2}$	γ^2	1	1
$\mathbf{1}_{1,0}$	1	γ	1
$\mathbf{1}_{1,1}$	γ	γ	1
$\mathbf{1}_{1,2}$	γ^2	γ	1
$\mathbf{1}_{2,0}$	1	γ^2	1
$\mathbf{1}_{2,1}$	γ	γ^2	1
$\mathbf{1}_{2,2}$	γ^2	γ^2	1
$\mathbf{3}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma & 0 \\ 1 & 0 & \gamma^2 \end{pmatrix}$	$\begin{pmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{pmatrix}$
$\bar{\mathbf{3}}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \gamma^2 & 0 \\ 1 & 0 & \gamma \end{pmatrix}$	$\begin{pmatrix} \gamma^2 & 0 & 0 \\ 0 & \gamma^2 & 0 \\ 0 & 0 & \gamma^2 \end{pmatrix}$

Table 3: The eleven irreducible representations of $\Delta(27)$. Here $\gamma = e^{2\pi i/3}$ is a cube root of unity.

The multiplication table for the above representations is given by

$\otimes$	$\mathbf{1}_{i,j}$	$\mathbf{3}$	$\bar{\mathbf{3}}$
$\mathbf{1}_{k,l}$	$\mathbf{1}_{i+k,j+l}$	$\mathbf{3}$	$\bar{\mathbf{3}}$
$\mathbf{3}$	$\mathbf{3}$	$\bar{\mathbf{3}}_{(\bar{1})} \oplus \bar{\mathbf{3}}_{(\bar{2})} \oplus \bar{\mathbf{3}}_{(\bar{3})}$	$\oplus_{i,j} \mathbf{1}_{i,j}$
$\bar{\mathbf{3}}$	$\bar{\mathbf{3}}$	$\oplus_{i,j} \mathbf{1}_{i,j}$	$\mathbf{3}_{(1)} \oplus \mathbf{3}_{(2)} \oplus \mathbf{3}_{(3)}$

(B.3)

For our purposes, we desire the exact form of the superpotential, which may be obtained from the appropriate set of Clebsch-Gordan coefficients. In particular, we will need the coefficients in the decomposition $\mathbf{3} \otimes \mathbf{r}_i \to \mathbf{r}_j$. We will label the functions that $\mathbf{1}_{i,j}$ work on as $\phi_{i,j}$ and the functions that $\mathbf{3}$ work on as triplets Λ_j. If multiple representations of the same kind appear in a product or sum of representations, an index in parentheses will appear on both the representation as well as the functions associated with it. The functions transforming under a barred representation will simply be labeled by putting a bar over the functions (although one should not take this as complex conjugation). In the following, we show the direct product and its resultant sum. The combinations of functions from the product that the resultant works on in the way shown above is displayed after the colon. For tensor products of the one-dimensional representations with **3**, we have

$$\begin{aligned}
\mathbf{3}\otimes\mathbf{1}_{0,0} &= \mathbf{3}:\begin{pmatrix}\Lambda_1\phi_{0,0}\\ \Lambda_2\phi_{0,0}\\ \Lambda_3\phi_{0,0}\end{pmatrix}, \\
\mathbf{3}\otimes\mathbf{1}_{0,1} &= \mathbf{3}:\begin{pmatrix}\Lambda_1\phi_{0,1}\\ \gamma\Lambda_2\phi_{0,1}\\ \gamma^2\Lambda_3\phi_{0,1}\end{pmatrix}, \quad \mathbf{3}\otimes\mathbf{1}_{0,2}=\mathbf{3}:\begin{pmatrix}\Lambda_1\phi_{0,2}\\ \gamma^2\Lambda_2\phi_{0,2}\\ \gamma\Lambda_3\phi_{0,2}\end{pmatrix}, \\
\mathbf{3}\otimes\mathbf{1}_{1,0} &= \mathbf{3}:\begin{pmatrix}\Lambda_3\phi_{1,0}\\ \Lambda_1\phi_{1,0}\\ \Lambda_2\phi_{1,0}\end{pmatrix}, \quad \mathbf{3}\otimes\mathbf{1}_{2,0}=\mathbf{3}:\begin{pmatrix}\Lambda_2\phi_{2,0}\\ \Lambda_3\phi_{2,0}\\ \Lambda_1\phi_{2,0}\end{pmatrix}, \\
\mathbf{3}\otimes\mathbf{1}_{1,1} &= \mathbf{3}:\begin{pmatrix}\gamma^2\Lambda_3\phi_{1,1}\\ \Lambda_1\phi_{1,1}\\ \gamma\Lambda_2\phi_{1,1}\end{pmatrix}, \quad \mathbf{3}\otimes\mathbf{1}_{2,2}=\mathbf{3}:\begin{pmatrix}\Lambda_2\phi_{2,2}\\ \gamma^2\Lambda_3\phi_{2,2}\\ \gamma\Lambda_1\phi_{2,2}\end{pmatrix}, \\
\mathbf{3}\otimes\mathbf{1}_{2,1} &= \mathbf{3}:\begin{pmatrix}\gamma\Lambda_2\phi_{2,1}\\ \gamma^2\Lambda_3\phi_{2,1}\\ \Lambda_1\phi_{2,1}\end{pmatrix}, \quad \mathbf{3}\otimes\mathbf{1}_{1,2}=\mathbf{3}:\begin{pmatrix}\Lambda_3\phi_{1,2}\\ \gamma^2\Lambda_1\phi_{1,2}\\ \gamma\Lambda_2\phi_{1,2}\end{pmatrix}.
\end{aligned} \tag{B.4}$$

Similarly, one can work out how the one-dimensional representations act on the $\bar{\mathbf{3}}$ representation

$$\begin{aligned}
\bar{\mathbf{3}}\otimes\mathbf{1}_{0,0} &= \bar{\mathbf{3}}:\begin{pmatrix}\Lambda_1\phi_{0,0}\\ \Lambda_2\phi_{0,0}\\ \Lambda_3\phi_{0,0}\end{pmatrix}, \\
\bar{\mathbf{3}}\otimes\mathbf{1}_{0,1} &= \bar{\mathbf{3}}:\begin{pmatrix}\Lambda_1\phi_{0,1}\\ \gamma\Lambda_2\phi_{0,1}\\ \gamma^2\Lambda_3\phi_{0,1}\end{pmatrix}, \quad \bar{\mathbf{3}}\otimes\mathbf{1}_{0,2}=\bar{\mathbf{3}}:\begin{pmatrix}\Lambda_1\phi_{0,2}\\ \gamma^2\Lambda_2\phi_{0,2}\\ \gamma\Lambda_3\phi_{0,2}\end{pmatrix}, \\
\bar{\mathbf{3}}\otimes\mathbf{1}_{1,0} &= \bar{\mathbf{3}}:\begin{pmatrix}\Lambda_2\phi_{1,0}\\ \Lambda_3\phi_{1,0}\\ \Lambda_1\phi_{1,0}\end{pmatrix}, \quad \bar{\mathbf{3}}\otimes\mathbf{1}_{2,0}=\bar{\mathbf{3}}:\begin{pmatrix}\Lambda_3\phi_{2,0}\\ \Lambda_1\phi_{2,0}\\ \Lambda_2\phi_{2,0}\end{pmatrix}, \\
\bar{\mathbf{3}}\otimes\mathbf{1}_{1,1} &= \bar{\mathbf{3}}:\begin{pmatrix}\gamma\Lambda_2\phi_{1,1}\\ \gamma^2\Lambda_3\phi_{1,1}\\ \Lambda_1\phi_{1,1}\end{pmatrix}, \quad \bar{\mathbf{3}}\otimes\mathbf{1}_{2,2}=\bar{\mathbf{3}}:\begin{pmatrix}\gamma\Lambda_3\phi_{2,2}\\ \Lambda_1\phi_{2,2}\\ \gamma^2\Lambda_2\phi_{2,2}\end{pmatrix}, \\
\bar{\mathbf{3}}\otimes\mathbf{1}_{2,1} &= \bar{\mathbf{3}}:\begin{pmatrix}\gamma^2\Lambda_3\phi_{2,1}\\ \Lambda_1\phi_{2,1}\\ \gamma\Lambda_2\phi_{2,1}\end{pmatrix}, \quad \bar{\mathbf{3}}\otimes\mathbf{1}_{1,2}=\bar{\mathbf{3}}:\begin{pmatrix}\gamma^2\Lambda_2\phi_{1,2}\\ \gamma\Lambda_3\phi_{1,2}\\ \Lambda_1\phi_{1,2}\end{pmatrix}.
\end{aligned} \tag{B.5}$$

Next, we display similarly the product of the $\mathbf{3}$ with itself:

$$\begin{aligned}
\mathbf{3}^{(1)} \otimes \mathbf{3}^{(2)} &= \bar{\mathbf{3}}_{(\bar{1})} \oplus \bar{\mathbf{3}}_{(\bar{2})} \oplus \bar{\mathbf{3}}_{(\bar{3})}; \\
\bar{\mathbf{3}}_{(\bar{1})} &: \begin{pmatrix} \Lambda_1^{(1)}\Lambda_1^{(2)} \\ \Lambda_2^{(1)}\Lambda_2^{(2)} \\ \Lambda_3^{(1)}\Lambda_3^{(2)} \end{pmatrix}, \\
\bar{\mathbf{3}}_{(\bar{2})} &: \begin{pmatrix} \Lambda_2^{(1)}\Lambda_3^{(2)} \\ \Lambda_3^{(1)}\Lambda_1^{(2)} \\ \Lambda_1^{(1)}\Lambda_2^{(2)} \end{pmatrix}, \\
\bar{\mathbf{3}}_{(\bar{3})} &: \begin{pmatrix} \Lambda_3^{(1)}\Lambda_2^{(2)} \\ \Lambda_1^{(1)}\Lambda_3^{(2)} \\ \Lambda_2^{(1)}\Lambda_1^{(2)} \end{pmatrix},
\end{aligned} \tag{B.6}$$

and of $\mathbf{3}$ with $\bar{\mathbf{3}}$

$$\begin{aligned}
\mathbf{3} \otimes \bar{\mathbf{3}} &= \oplus_{i,j} \mathbf{1}_{i,j}; \\
\mathbf{1}_{0,0} &: \frac{1}{\sqrt{3}} \left(\Lambda_1 \bar{\Lambda}_{\bar{1}} + \Lambda_2 \bar{\Lambda}_{\bar{2}} + \Lambda_3 \bar{\Lambda}_{\bar{3}} \right), \\
\mathbf{1}_{0,1} &: \frac{1}{\sqrt{3}} \left(\gamma \Lambda_1 \bar{\Lambda}_{\bar{1}} + \Lambda_2 \bar{\Lambda}_{\bar{2}} + \gamma^2 \Lambda_3 \bar{\Lambda}_{\bar{3}} \right), \\
\mathbf{1}_{0,2} &: \frac{1}{\sqrt{3}} \left(\gamma^2 \Lambda_1 \bar{\Lambda}_{\bar{1}} + \Lambda_2 \bar{\Lambda}_{\bar{2}} + \gamma \Lambda_3 \bar{\Lambda}_{\bar{3}} \right), \\
\mathbf{1}_{1,0} &: \frac{1}{\sqrt{3}} \left(\Lambda_1 \bar{\Lambda}_{\bar{3}} + \Lambda_2 \bar{\Lambda}_{\bar{1}} + \Lambda_3 \bar{\Lambda}_{\bar{2}} \right), \\
\mathbf{1}_{2,0} &: \frac{1}{\sqrt{3}} \left(\Lambda_1 \bar{\Lambda}_{\bar{2}} + \Lambda_2 \bar{\Lambda}_{\bar{3}} + \Lambda_3 \bar{\Lambda}_{\bar{1}} \right), \\
\mathbf{1}_{1,1} &: \frac{1}{\sqrt{3}} \left(\gamma \Lambda_1 \bar{\Lambda}_{\bar{3}} + \Lambda_2 \bar{\Lambda}_{\bar{1}} + \gamma^2 \Lambda_3 \bar{\Lambda}_{\bar{2}} \right), \\
\mathbf{1}_{2,2} &: \frac{1}{\sqrt{3}} \left(\gamma \Lambda_1 \bar{\Lambda}_{\bar{3}} + \gamma^2 \Lambda_2 \bar{\Lambda}_{\bar{1}} + \gamma \Lambda_3 \bar{\Lambda}_{\bar{2}} \right), \\
\mathbf{1}_{2,1} &: \frac{1}{\sqrt{3}} \left(\gamma \Lambda_1 \bar{\Lambda}_{\bar{2}} + \Lambda_2 \bar{\Lambda}_{\bar{3}} + \gamma^2 \Lambda_3 \bar{\Lambda}_{\bar{1}} \right), \\
\mathbf{1}_{1,2} &: \frac{1}{\sqrt{3}} \left(\Lambda_1 \bar{\Lambda}_{\bar{2}} + \gamma \Lambda_2 \bar{\Lambda}_{\bar{3}} + \gamma^2 \Lambda_3 \bar{\Lambda}_{\bar{1}} \right).
\end{aligned}$$

In addition to these, we always take the Clebsch-Gordan coefficients for the product of two one-dimensional representations to be simply 1. Also, the Clebsch-Gordan coefficient associated with the product $\mathbf{r}\otimes\bar{\mathbf{r}} \to \mathbf{1}_{0,0}$ is always taken to be $(1/\sqrt{\dim \mathbf{r}})\sum f_i \bar{f}_i$ in the usual way.

We take the expressions for the Clebsch-Gordan coefficients to be commutative if two different representations are commutative, even though the Kronecker product of two matrices is not. Note that the above Clebsch-Gordon coefficients are defined only up to a phase for each representation in the sum. This means that, although we have displayed above that the phases of the $\mathbf{1}_{0,1}$ and $\overline{(\mathbf{1}_{0,1})} = \mathbf{1}_{0,2}$ representations appearing in $\mathbf{3} \otimes \bar{\mathbf{3}}$ are related, they are in fact not related; we can independently rephase the two representations. This corresponds directly to redefining the fields with respect to their phase. We have chosen these phases to make a certain symmetry manifest in the resulting field theory.

We make one final note on this choice of Clebsch-Gordan coefficients. While the Kronecker product of three matrices is associative, the above choice for Clebsch-Gordan coefficients is not. However, this does not affect the physics. The above prescriptions define how to construct singlets out of product representations. Any method for arriving at these will be physically equivalent, as the number of linearly independent singlet functions is fixed for a given product. Therefore, the only possibility is that two different conventions are related by some unitary transformation (taking that both conventions involve unitary Clebsch-Gordan coefficients).

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